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SEPARABLE ABELIAN p-GROUPS HAVING CERTAIN PRESCRIBED CHAINS

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ABSTRACT

An abelian p-group G is called $p^{\omega+1}$-projective if $p^{\omega+1}\mathrm{Ext}(G,X) = 0$ for all groups X. This class of groups constitutes a natural extension of the well-known class of totally projective groups whose members are precisely those groups classifiable by the Ulm–Kaplansky invariants. Fuchs asked whether $p^{\omega+1}$-projective groups G can be characterized in terms of filtrations of G. Our Theorem 1 provides counterexamples.

§1. Results

Totally projective groups constitute the largest class of abelian p-groups which can be classified by Ulm–Kaplansky invariants, cf. Fuchs [5]. This result attracted special attention to this class of p-groups and its natural extensions. Hence it is natural to consider p^σ-projective p-groups G having the property that $p^\sigma\mathrm{Ext}(G,X) = 0$ for all groups X, cf. Fuchs [5, p. 89]. Recall that G is totally projective if and only if $p^\sigma\mathrm{Ext}(G/p^\sigma G,X) = 0$ for all groups X and for all ordinals σ; cf. Fuchs [5, p. 89].

The class of p^σ-projective p-groups has been investigated for particular ordinals σ in [2, 3, 6, 8, 9] and in papers mentioned there. It is easy to see that p^ω-projective p-groups are direct sums of cycles. Nunke [8] proved a more general result that G is $p^{\omega+n}$-projective if and only if G/P is a direct sum of cyclic groups for some subgroup $P \subseteq G[p^n]$ of the socle $G[p^n] = \{g \in G : p^n g = 0\}$.

Fuchs asked whether $p^{\omega+1}$-projective groups could be characterized in terms of filtrations (= continuous ascending chains of subgroups terminating at G), cf. [2, p. 43]. In [3] it was noted that every $p^{\omega+1}$-projective group G of cardinality κ possesses a κ-filtration $G = \bigcup_{\alpha<\kappa} G_\alpha$ (with $|G_\alpha| < \kappa$) such that $p^{\omega+1}(G/G_\alpha) = 0$ for all $\alpha < \kappa$.

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We want to show that the converse does not hold. In fact, we will prove the following much stronger

THEOREM 1. *Let R be a ring with additive structure R^+ such that*

$$\bigoplus_{\kappa} J_p \subseteq R^+ \subseteq \widehat{\bigoplus_{\kappa} J_p}$$

where $J_p = \hat{\mathbf{Z}}$ denotes the p-adic integers and $\hat{A}$ denotes the p-adic completion of the (p-reduced) abelian group A.

For any cardinal $\lambda = \lambda^{\aleph_0} > |R|$ we can find a separable abelian p-group $G = \bigcup_{\alpha<\lambda} G_\alpha$ with a continuous chain of subgroups $\{G_\alpha : \alpha < \lambda\}$ containing 0 such that the following hold:

(i) $p^{\omega+1}(G/G_\alpha) = 0$,

(ii) *G is not p^σ-projective for any ordinal σ,*

(iii) $|G| = \lambda$,

(iv) $\operatorname{End} G = R + \operatorname{Small}^{[p]} G$ *where* $\operatorname{Small}^{[p]} G$ *is the ideal*

$$\{\varphi \in \operatorname{End} G : \exists n \text{ such that } (p^n G[p])\varphi = 0\}$$

of all endomorphisms of G which are "small" on the socle

$$G[p] = \{a \in G : pa = 0\} \text{ of } G,$$

(v) $R \cap \operatorname{Small}^{[p]} G = pR$.

Assuming GCH, we can find a proper class of regular cardinals λ such that G with $|G| = \lambda$ (as in Theorem 1) has a λ-filtration $G = \bigcup_{\alpha<\lambda} G_\alpha$; apply the construction and some set theoretic arguments from [4]. Assuming CH, weak diamond holds and a similar argument as under $V = L$ gives an ω_1-filtration $G = \bigcup_{\alpha<\omega_1} G_\alpha$ of a group G of cardinal $\lambda = \omega_1$ as in Theorem 1. The free parameter R in Theorem 1 can be specified to prescribe decomposition properties of G, which are similar to [1]. Here we concentrate on a special case.

COROLLARY 2. *Let G and R be as in Theorem 1 and assume that the ring R/pR has only the trivial idempotents 0 and 1. Then G is essentially indecomposable.*

Recall that G is essentially indecomposable (in the sense of R. S. Pierce) if any direct decomposition of G involves a bounded summand.

PROOF. If $G = A \oplus C$ and $\pi : G \to A$ is the canonical projection, then $\pi^2 = \pi \in \operatorname{End} G$. From Theorem 1(iv) we derive $\pi = r + s$ for some $r \in R$ and $s \in \operatorname{Small}^{[p]} G \lhd \operatorname{End} G$. Hence $(r+s)^2 = r + s$ implies $r - r^2 \in R \cap \operatorname{Small}^{[p]} G$. Theorem 1(v) implies $r - r^2 \in pR$ and $\bar{r} = r + pR$ is an idempotent of R/pR. We derive $\bar{r} = 0$ or $\bar{r} = 1$ from our assumption on R/pR. If $\bar{r} = 0$, then $r \in pR$ and $\pi = r + s \in \mathrm{Small}^{[p]} G$. If $\bar{r} = 1$, then $1 - r - s \in \mathrm{Small}^{[p]} G$ and we may assume $\pi = \pi^2 \in \mathrm{Small}^{[p]} G$. We derive $(p^n G[p])\pi = 0$ which implies $p^n A = 0$ and A is bounded.

Assuming $V = L$, then Corollary 2 (without Theorem 1(ii)) for regular cardinals λ is due to Cutler, Mader, Megibben [2]. Their proof is based on an "antique" method due to Eklof and Mekler from 1977 for constructing indecomposable abelian groups in L, cf. [2] or [4] for references. It is the aim of this paper to give a simpler proof of a stronger result which also holds in ordinary set theory of ZFC. Moreover, we ensure that our group G is not p^σ-projective for any σ, a question which could not be decided in [2].

§2. Construction of certain separable p-groups

The proof of Theorem 1 will use a combinatorial idea due to Shelah [10]. An elementary proof of this is given in the appendix of Corner, Göbel [1], and we refer to this as (Shelah's) Back Box [1]. In order to apply the Black Box we have to set up some notation and state the combinatorial result.

(a) *Algebraic setup*

Let R be the ring and λ be the cardinal given in Theorem 1. Hence

$$(*) \qquad \bigoplus_{\alpha \in \kappa} e_\alpha J_p \subseteq R^+ \subseteq \widehat{\bigoplus_{\alpha \in \kappa} e_\alpha R}$$

where e_ν ($\nu \in \kappa$) labels the components of the direct sum and $e_0 = 1 \in R$. Let $T = {}^{\omega >}\lambda$ denote the tree of all functions $\tau : n \to \lambda$ ($n < \omega$) ordered by set theoretical containment. For $\tau \in T$, define the length $l(\tau)$ to be the domain $\mathrm{dom}(\tau) = n$.

If $\tau \in T$ and $l(\tau) = n$, then τ is considered as generator of the cyclic R-module τR with annihilator $\mathrm{Ann}_R(\tau) = p^n R$, i.e.

$$\tau R = \bigoplus_{\nu \in \kappa} e_{\tau\nu} J_p$$

with $e_{\tau\nu} J_p \cong J_p / p^n J_p$ canonically.

Finally, let

$$B = \bigoplus_{\tau \in T} \tau R$$

and $\bar{B}$ its torsion-completion which is the torsion subgroup of the p-adic completion $\hat{B}$ of B.

(b) *Combinatorial setup*

Every $g \in \bar{B}$ is expressible as a convergent sum $g = \sum_{\tau \in T} \tau g_\tau$ with $g_\tau \in R/p^n R$. The support of g is defined to be

$$[g] = \{\tau \in T : g_\tau \neq 0\},$$

which can obviously be extended to subsets of $\bar{B}$. Observe that $|[X]| \leq |X| \cdot \aleph_0$ for any subset $X \subseteq \bar{B}$, which is crucial. In order to include all singular cardinals λ of cofinality $> \omega$ into Theorem 1, we define a norm $\| \; \| : \mathrm{cf}(\lambda) + 1 \to \lambda + 1$ which is any fixed, continuous, strictly increasing function with $\|0\| = 0$ and $\|\mathrm{cf}(\lambda)\| = \lambda$. This can be extended to T and subsets of $\bar{B}$, e.g.

$$\|g\| = \min\{\nu \leq \mathrm{cf}(\lambda) : [g] \in {}^{\omega>}\|\nu\|\},$$

$$\|X\| = \sup\{\|x\| : x \in X\} \qquad \text{for any } X \subseteq \bar{B}.$$

If $\|X\|$ is undefined, we say $\|X\| = \infty$, and this can only happen if $|X| \geq \mathrm{cf}(\lambda) > \aleph_0$ by $\lambda^{\aleph_0} = \lambda$ and König's theorem. R-submodules generated by countable subsets of T are called canonical submodules. A trap is a triple (f, P, φ) where f: ${}^{\omega>}\omega \to T$ is a tree embedding, P is a canonical submodule of B and $\varphi \in \mathrm{End}\, \bar{P}$ satisfying the following conditions:

(i) $\mathrm{Im} f \subseteq P$,

(ii) $[P] \subseteq P$ and $[P]$ is a subtree of T,

(iii) $\mathrm{cf}(\|P\|) = \omega$,

(iv) $\|v\| = \|P\|$ for all branches v of $\mathrm{Im} f$.

Recall that a branch v is a map $v : \omega \to \lambda$ which will be identified with the maximal linear set $\{v \restriction n : n \in \omega\}$ of T. Let $\mathrm{Br}(\ldots)$ denote all branches in.... A constant branch is a branch v in T with $v : \omega \to \{\alpha\} \subset T$ for some ordinal $\alpha < \lambda$.

Now we can state

Shelah's Black Box. *For some ordinal λ^* of cardinality λ there exists a transfinite sequence of traps $(f_\alpha, P_\alpha, \varphi_\alpha)$ $(\alpha < \lambda^*)$ such that for $\alpha, \beta < \lambda^*$,*

(a) $\beta < \alpha \Rightarrow \|P_\beta\| \leq \|P_\alpha\|$,

(b) $\beta \neq \alpha \Rightarrow \mathrm{Br}(\mathrm{Im} f_\alpha) \cap \mathrm{Br}(\mathrm{Im} f_\beta) = \varnothing$,

(c) $\beta + 2^{\aleph_0} \leq \alpha \Rightarrow \mathrm{Br}(\mathrm{Im} f_\alpha) \cap \mathrm{Br}([P_\beta]) = \varnothing$,

(d) *for any countable subset X of $\bar{B}$ and any $\varphi \in \mathrm{End}\, \bar{B}$ there exists $\alpha < \lambda^*$ such that*

$$X \subseteq \overline{P_\alpha}, \quad \|X\| \leq \|P_\alpha\|, \quad \varphi \restriction P_\alpha = \varphi_\alpha.$$

(c) *Construction of the group G*

Let $(f_\alpha, P_\alpha, \varphi_\alpha)$ $(\alpha < \lambda^*)$ be the sequence of traps on $B = \bigoplus_T \tau R$ given by the Black Box. In order to make sure that the outcoming group is not p^σ-projective we use the constant branch o with $o(n) = 0$ for a special pure subgroup H of G. Observe that $\bigoplus_{\tau \in o} \tau R_p$ has a canonical summand $D \cong \bigoplus_{i \in \omega} Z_{p^i}$. Moreover $\|D\| = 0$ is minimal.

Take any thin group H such that $D \subseteq H \subseteq \bar{D}$ is pure and $|H| > \aleph_0$. Hence D is a countable basic subgroup of H and H is not "subprojective" (a subgroup of a totally projective group), cf. L. Salce [9, p. 186, Theorema 36.10]. Such a group H cannot be p^σ-projective for any ordinal σ. Recall that the separable group H is thin if and only if all homomorphisms from $\bar{D}$ into H are small—which will be used in the proof of (iv) in Theorem 1.

We will construct G as the union of a continuous chain $\{G_\alpha : \alpha < \lambda^*\}$ of thin subgroups G_α of $\bar{B}$ subject to the following conditions:

$$G_0 = H \quad \text{and} \quad G_1 = \langle H, b : b \in B, \|b\| < \|P_1\| \rangle.$$

Let $\mu \le \lambda^*$, and assume that G_α $(\alpha < \mu)$ have been constructed.

If μ is a limit, then $G_\mu = \bigcup_{\alpha < \mu} G_\alpha$ is defined by continuity. When $\mu = \alpha + 1$ is a successor, we distinguish cases, based on the following conditions:

The Black Box provides a trap $(f_\alpha, P_\alpha, \varphi_\alpha)$.

If $\|P_{\alpha+1}\| > \|P_\alpha\|$, then we first replace G_α by an extension

$$G_\alpha^* = \langle G_\alpha, b \in B : \|b\| < \|P_{\alpha+1}\| \rangle.$$

Clearly G_α is a direct summand of G_α^* with Σ-cyclic quotient. Abusing notation we will identify G_α and G_α^*.

Then we pick a constant branch $k_\alpha = \{k_\alpha(i) : i \in \omega\}$ with $l(k_\alpha(i)) = i$ such that $\|k_\alpha(i)\| > \|P_{\alpha+1}\|$ for all $i \in \omega$ and $[k_\alpha] \cap \bigcup_{\beta < \alpha} [k_\beta] = \emptyset$. This is possible by cardinality reasons.

Consider

$$a = \sum_{i \in \omega} p^{l(g_\alpha(i)) - 1} g_\alpha(i) \quad \text{and} \quad y_n = \sum_{i \ge n} p^{l(k_\alpha(i)) - n} k_\alpha(i) \quad \text{for all } n \in \omega.$$

Then $a \in \bar{B}[p]$ and y_n has order p^n. Moreover, $\{y_n p^{n-1} : n \in \omega \setminus \{0\}\}$ is independent mod G_α over R.

If

$$b_n = \sum_{i=1}^{n} p^{l(g_\alpha(i))} g_\alpha(i) \quad \text{and} \quad a_n = \sum_{i=n+1}^{\infty} p^{l(g_\alpha(i)) - n - 1} g_\alpha(i),$$

then $a = p^n a_n + b_n$ for $n \in \omega$, and if $x_n = a_n + y_n$ then $p^n x_n = a - b_n$ for all $n \in \omega$.

Now we define

(i)
$$G_{\alpha+1} = \langle G_\alpha, g_{\alpha n}R : n \in \omega \rangle \subseteq \bar{B}$$

where

(ii)
$$g_{\alpha n} = x_n, \quad a_\alpha = a, \quad b_{\alpha n} = b_n \qquad (n \in \omega).$$

Note that $p^{\omega+1}(G_{\alpha+1}/G_\alpha) = 0$ and that

$$G_{\alpha+1}/\langle k_\beta(i)R, b : \| b \| < \| P_\alpha \|, \beta \leq \alpha, i \in \omega \rangle$$

is a divisible p-group. Moreover, $k_\alpha(i)R \subset G_{\alpha+1}$ for all $i \in \omega$.

Some elementary computations show that if $s \in G_{\alpha+1}[p]$, then $\{\tau \in [s], \tau > \| P_{\alpha+1} \| \}$ is finite.

We will use $G_{\alpha+1}$ with (i) and (ii) for the next step in the construction of G, provided the $g_{\alpha n}$'s are "best possible" in the following sense.

Recall that φ_α has a unique extension $\bar{\varphi}_\alpha : \bar{P}_\alpha \to \bar{B}$ and $a_\alpha \in \bar{P}_\alpha$. Hence either $a_\alpha \bar{\varphi}_\alpha \in G_{\alpha+1}$ or $a_\alpha \bar{\varphi}_\alpha \notin G_{\alpha+1}$ and we consider (ii) to be "best possible"

if $a_\alpha \bar{\varphi}_\alpha \notin G_{\alpha+1}$

and

(*) if $a_\beta \bar{\varphi}_\beta \notin G_\alpha$ for some $\beta < \alpha$, then $a_\beta \bar{\varphi}_\beta \notin G_{\alpha+1}$ as well.

Observe that G_α does not contain a basic subgroup of $G_{\alpha+1}$ in general, hence any homomorphism $\varphi : G_\alpha \to \bar{B}$ may have many extensions $\psi : G_{\alpha+1} \to \bar{B}$. However, all G_α are pure subgroups of $\bar{B}$ since $G_\alpha/(G_\alpha \cap B)$ is divisible. Despite this fact, we use only one extension, $\bar{\varphi} \restriction G_{\alpha+1}$, for controlling φ.

If the first choice of $G_{\alpha+1}$ and $g_{\alpha n}$, respectively, is not possible, we will work with another extension $G_{\alpha+1}$ of G_α as in (i) and we will use new elements $g_{\alpha n} \in \bar{B}[p^{n+1}]$ defined as follows.

Find $s_n \in \bar{P}_\alpha[p^{n+1}]$ $(n \in \omega)$ such that

(iii)
$$s_0 = s \quad \text{and} \quad s - p^n s_n = -b'_n \in G_\alpha \quad (n \in \mathbf{N}),$$

(iv)
$$\sup_{n \in \omega} \| s_n \| < \| \bar{P}_\alpha \|,$$

and define

(ii)*
$$g_{\alpha n} = x_n + s_n, \quad a_\alpha = a + s, \quad b_{\alpha n} = b_n - b'_n$$

and $G_{\alpha+1}$ as in (i).

In any case we have

$$\text{(v)} \qquad p^n g_{\alpha n} = a_\alpha - b_{\alpha n} \qquad \text{with } b_{\alpha n} \in G_\alpha.$$

If $a_\alpha \bar{\varphi}_\alpha \notin G_{\alpha+1}$ and (*) holds, we will use (ii)* as our second choice.

If this also is not possible, we finally stick to our first choice (ii) and do not require $a_\alpha \bar{\varphi}_\alpha \notin G_{\alpha+1}$ any more.

We set $G = \bigcup_{\alpha<\lambda^*} G_\alpha$ and observe that G is a pure subgroup of $\bar{B}$ containing B.

We have two immediate consequences of the Black Box and the construction of G.

Proposition 2.1. *Let $G_{\alpha+1} = \langle G_\alpha, g_{\alpha n} R : n \in \omega \rangle$, a_α as above and let $z_n \in \mathbf{Z}$. Then the following hold*:

(a) *$\Sigma g_{\alpha n} z_n \in \langle G_\alpha, a_\alpha R \rangle$ if and only if $p^n \mid z_n$ for all n.*

(b) *$\Sigma g_{\alpha n} z_n \in G_\alpha$ if and only if $p^n \mid z_n$ for all n and $p \mid \Sigma z_n p^{-n}$.*

(c) *If $\bar{a}_\alpha = a_\alpha + G_\alpha$ and $\bar{g}_{\alpha n} = g_{\alpha n} + G_\alpha$, then*

$$G_{\alpha+1}/G_\alpha = \langle \bar{a}_\alpha R, \bar{g}_{\alpha n} R : n \in \omega \rangle \cong \bigoplus H_{\omega+1}.$$

(d) *$p^{\omega+1}(G/G_{\alpha+1}) \subseteq \langle \bar{a}_\beta R : \alpha \leq \beta < \alpha + 2^{\aleph_0} \rangle$ is an elementary abelian p-group of rank at most $2^{\aleph_0}$.*

Remark. $H_{\omega+1}$ is the (generalized) Prüfer group of length $\omega + 1$, cf. Fuchs [5, pp. 85, 86].

Proof. (a) If $p^n \mid z_n$, then clearly $\Sigma g_{\alpha n} z_n \in \langle G_\alpha, a_\alpha R \rangle$.

Conversely let $\Sigma g_{\alpha n} z_n \in \langle G_\alpha, a_\alpha R \rangle$. Note that $a_\alpha \in \bar{G}_\alpha$ and compute

$$\Sigma g_{\alpha n} z_n \equiv \Sigma y_n z_n \bmod \bar{G}_\alpha$$

which is $\equiv 0$ only if $p^n \mid z_n$ by the choice of a constant branch k_α.

(b) is similar to (a).

(c) Since R^+ is the p-adic completion of a direct sum of copies of the additive group J_p of p-adic integers, we may assume that $R^+ = J_p$ and want to show that $G_{\alpha+1}/G_\alpha \cong H_{\omega+1}$. The generalized Prüfer group $H_{\omega+1}$ is defined by generators $\langle a, b_i : i \in \omega \rangle$ with $p^\omega H_{\omega+1} = \langle a \rangle \cong Z_p$ and $H_{\omega+1}/\langle a \rangle \cong \bigoplus_{i\in\omega} \langle b_i + \langle a \rangle \rangle$, cf. Fuchs [5, p. 85]. The identification ($a \to \bar{a}_\alpha$, $b_i \to \bar{g}_{\alpha i}$) gives rise to the desired isomorphism.

(d) follows immediately from (c) and the construction of G.

Proposition 2.2. *Condition (*) in the construction is automatically satisfied.*

PROOF. Corner, Göbel [1, p. 458, Corollary 3.10]. Observe that the nontrivial part of the two-line proof of Corollary 3.10 in [1] is hidden in an application of Lemma 3.9 from [1].

§3. Proof of Theorem 1.

We want to show that the group G constructed in §2 satisfies all conditions of Theorem 1.

G is a pure subgroup of $\bar{B}$ containing B. Hence it is immediate that G is a separable, abelian p-group of cardinality λ. It contains a pure subgroup H of cardinality $> \aleph_0$ with countable basis. Hence G is not p^σ-projective for any ordinal σ, cf. Salce [9, p. 186]. From Proposition 2.1(d) we derive

$$p^{\omega+1}(G/G_\alpha) \subseteq p\langle \bar{a}_\beta : \alpha \leq \beta < \alpha + 2^{\aleph_0}\rangle_R = 0.$$

It remains to show (iv) and (v).

The ring R acts faithfully on B by scalar multiplication.

Moreover, $B \subseteq G$ are R-modules, and we will identify $R \subseteq \operatorname{End} G$, hence

$$R + \operatorname{Small}^{[p]} G \subseteq \operatorname{End} G.$$

By way of contradiction, let $\varphi \in \operatorname{End} G \backslash R + \operatorname{Small}^{[p]} G$. The homomorphism $\varphi \restriction B$ has a unique extension $\bar{\varphi} \in \operatorname{End} \bar{B}$.

The following argument is similar to Corner, Göbel [1, pp. 471, 472 and 459, 460], however it also differs substantially because $G_{\alpha+1}$ is no longer contained in the p-adic closure of G_α.

For each $r \in R$ we have $\varphi - r \notin \operatorname{Small}^{[p]} G$ and we can choose $d_k \in p^k B[p]$ such that

$$0 \neq h_k = d_k(\varphi - r) \in p^k \bar{B}[p] \cap G \qquad \text{for all } k \in \omega.$$

We may also choose $\sigma_k \in [h_k]$ such that

(a) $\sup_{k\in\omega} \| \sigma_k \| = \sup_{k \in \omega} \| h_k \|$.

Passing to subsequences we may assume

(b) The sequences $\| h_k \|$ and $\| \sigma_k \|$ $(k \in \omega)$ are non-decreasing.

(c) $h_{k+1} \in p^{l(\sigma_n)} \bar{B}$.

(d) If infinitely many σ_k are on a branch of T, then all of them are.

It follows from (c) that all elements of $[h_{k+1}]$ are of greater length than $l(\sigma_k)$, hence $l(\sigma_k)$ is strictly increasing and the σ_k are all distinct.

Consider all converging sums $s = \Sigma \epsilon_k d_k \in \bar{B}$ $(\epsilon_k \in \{0,1\})$. Then the continuity argument in Corner, Göbel [1, p. 472] applies. We can find suitable $\epsilon_k \in \{0,1\}$ such that

(e) $s(\bar{\varphi} - r) \notin G$ where $s = s(r) \in \bar{B}[p]$.

We may assume $\epsilon_k = 1$ $(k \in \omega)$, hence $s = s_0 = \sum_{m \in \omega} p^m c_m$ for suitable $c_m \in B$. Moreover, let $s_n = \sum_{m \geq n} p^{m-n} c_m$ for $n \in \omega$. Next we will find

(f) $P \subseteq B$ a canonical summand such that $\bar{P}(\bar{\varphi} - r)$ is not contained in G for all $r \in R$.

Consider any canonical subgroup P_0 of $\bar{B}$ containing a constant branch element w. If (f) does not hold for P_0, there exist $r \in R$ with $\bar{P}_0(\bar{\varphi} - r) \subseteq G$. Take a canonical subgroup P containing P_0 such that $s = s(r) \in \bar{P}$ with $\|s\| < \|P\|$ and $s(\bar{\varphi} - r) \notin G$ from (e). If we can find $t \in R$ with $\bar{P}(\bar{\varphi} - t) \subseteq G$, then $\bar{P}_0(t - r) \subseteq G$ as well, and therefore $w(t - r) = 0$. This forces $s(\bar{\varphi} - r) = s(\underline{\varphi} - t) \in G$, contradicting (e). Now it is easy to improve (f).

(g) If P is as in (f), then $\bar{P}_o(p^k \bar{\varphi} - r)$ is not contained in G for all $r \in R$ with $r \notin pR$ or $p^k = 1$.

We may assume $k > 0$ by (f). Hence p is not a divisor of r and $\bar{P}[p](p^k \bar{\varphi} - r) \subset \bar{P}[p]r$ is not contained in G because of w.

Using the Black Box, we can find $\alpha < \lambda^*$ such that

(h) $P \subseteq \overline{P_\alpha}$, $\varphi \restriction \overline{P_\alpha} = \varphi_\alpha$,

and (g) and Proposition 2.2 ensure that the $g_{\alpha n}$ are first or second choice, hence

(k) $a_\alpha \bar{\varphi}_\alpha \notin G_{\alpha+1}$.

If $a_\alpha \varphi_\alpha \notin G_{\alpha+1}$, also $a_\alpha \bar{\varphi}_\alpha \notin G$ by (*) of the construction. Hence we may assume $a_\alpha \varphi_\alpha \in G_{\alpha+1}$. Next we show

(l) If $\psi \supset \varphi \restriction G_\alpha$ extends $\varphi \restriction G_\alpha$ such that $\operatorname{dom} \psi \supset G_{\alpha+1}$ and $a_\alpha \psi \in G_{\alpha+1}$, then $a_\alpha \psi = a_\alpha \bar{\varphi}$.

We have $p^{n+1} \mid (a_\alpha - b_{\alpha n})$ in G, hence $p^{n+1} \mid (a_\alpha \psi - b_{\alpha n} \varphi)$ and, similarly, $p^{n+1} \mid (a_\alpha \bar{\alpha} - b_{\alpha n} \varphi)$. We derive $p^{n+1} \mid (a_\alpha \psi - a_\alpha \bar{\varphi})$ in $\bar{B}$ and $a_\alpha \psi = a_\alpha \bar{\varphi}$ follows.

(m) If ψ is as in (l), then $a_\alpha \psi + G_\alpha \in p^\omega(G/G_\alpha)$.

Proof of (m). From $b_{\alpha n} \varphi \in G$ and $\|b_{\alpha n} \varphi\| < \|P_\alpha\|$ follows $b_{\alpha n} \varphi \in G_\alpha$, hence $p^{n+1} \mid (a_\alpha - b_{\alpha n}) \equiv a_\alpha \bmod G_\alpha$ and $p^{n+1} \mid a_\alpha \psi \bmod G_\alpha$. Proposition 2.1 implies $a_\alpha \psi + G_\alpha \in p^\omega(G/G_\alpha)$ and $a_\alpha \psi \equiv a_\alpha r \bmod G_\alpha$ for some $r \in R$.

Combining (h), (k) and (l) we have $a_\alpha \bar{\varphi}_\alpha \equiv a_\alpha r \bmod G_\alpha$, hence $a_\alpha \bar{\varphi}_\alpha \in G_{\alpha+1}$ contradicting (k).

We conclude $\operatorname{End} G = R + \operatorname{Small}^{[p]} G$ and (iv) follows.

Condition (v) of Theorem 1 follows immediately by construction of G. ∎

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